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Quantum Baxter-Belavin R-matrices and multidimensional Lax pairs for Painlevé VI

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Abstract

The quantum elliptic R-matrices of Baxter-Belavin type satisfy the associative Yang-Baxter equation in $\mathrm{Mat}(N,\mathbb{C})^{\otimes 3}$. The latter can be considered as noncommutative analogue of the Fay identity for the scalar Kronecker function. In this paper we extend the list of R-matrix valued analogues of elliptic function identities. In particular, we propose counterparts of the Fay identities in $\mathrm{Mat}(N,\mathbb{C})^{\otimes 2}$. As an application we construct R-matrix valued $2N^2 \times 2N^2$ Lax pairs for the Painlevé VI equation (in elliptic form) with four free constants using $\mathbb{Z}_N \times \mathbb{Z}_N$ elliptic R-matrix. More precisely, the four free constants case appears for an odd N while even N's correspond to a single constant.

Contents

1 Introduction and summary

In this paper we continue the study of identities for quantum (and classical) R-matrices, which are similar to the elliptic functions identities for scalar elliptic functions [13, 8]. More concretely, we prove the Fay identities in $\mathrm{Mat}(N,\mathbb{C})^{\otimes 2}$. It allows us to construct multidimensional Lax pairs for the Painlevé VI equation with the R-matrices as matrix elements.

We start with the list of properties and identities for elliptic functions, and then give their R-matrix version. Most of the properties are known from [2, 4], [14], [3, 15], [13] and [8].

Consider the following functions:

$$\phi(z,u)=\frac{\vartheta'(0)\vartheta(z+u)}{\vartheta(z)\vartheta(u)}\,, \tag{1.1}$$

$$E_1(z)=\frac{\vartheta'(z)}{\vartheta(z)}\,,\qquad E_2(z)=-\partial_z E_1(z)=\wp(z)-\frac{1}{3}\frac{\vartheta'''(0)}{\vartheta'(0)}\,, \tag{1.2}$$

where $\vartheta(z)$ is the odd Riemann theta-function

$$\vartheta(z)=\vartheta(z|\tau)=\sum_{k\in\mathbb{Z}}\exp\left(\pi\imath\tau(k+\frac{1}{2})^2+2\pi\imath(z+\frac{1}{2})(k+\frac{1}{2})\right) \tag{1.3}$$

and $\wp(z)$ is the Weierstrass $\wp$-function.

Following [16] the function (1.1) is referred to as the Kronecker function, and (1.2) are called the (first and the second) Eisenstein functions.

The Kronecker function can be considered as a section of the Poincaré bundle $\mathcal{P}$ over $\Sigma_\tau\times\Sigma'_\tau$. Here Σ_τ is the elliptic curve

$$\Sigma_\tau=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})\,,\quad \Im m\tau>0\,, \tag{1.4}$$

Σ'_τ – is its Jacobian $(\Sigma'_\tau\sim\Sigma_\tau)$. The Poincaré bundle $\mathcal{P}$ is a line bundle over $\Sigma_\tau\times\Sigma'_\tau$

$$
\begin{tikzcd}
 & \mathcal{P} \arrow[d] & \\
 & \Sigma_\tau \times \Sigma'_\tau \arrow[dl] \arrow[dr] & \\
\Sigma_\tau & & \Sigma'_\tau
\end{tikzcd}
\tag{1.5}
$$

specialized by (1.6), (1.7), (1.10) and (1.11).

The properties of theta-function (1.3) (including Riemann identities, see [11]) provides the following set of properties and relations for the functions (1.1)-(1.2):

- *Arguments symmetry:*

$$
\phi(z,u) = \phi(u,z)\,, \quad z \in \Sigma_\tau\,,\ u \in \Sigma'_\tau\,, \tag{1.6}
$$

- *Local expansion:*

$$
\phi(z,u) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \wp(u)) + O(z^2)\,, \tag{1.7}
$$

- *Residues:*

$$
\operatorname*{Res}_{z=0} \phi(z,u) = \operatorname*{Res}_{u=0} \phi(z,u) = \operatorname*{Res}_{z=0} E_1(z) = 1\,, \tag{1.8}
$$

- *Parity:*

$$
\phi(-z,-u) = -\phi(z,u)\,, \quad E_1(-z) = -E_1(z)\,, \quad E_2(-z) = E_2(z)\,, \tag{1.9}
$$

- *(Quasi)periodicity properties:*

$$
\phi(z+1,u) = \phi(z,u)\,, \quad E_1(z+1) = E_1(z)\,, \quad E_2(z+1) = E_2(z)\,, \tag{1.10}
$$

$$
\phi(z+\tau,u) = e^{-2\pi\imath u}\phi(z,u)\,, \quad E_1(z+\tau) = E_1(z) - 2\pi\imath\,, \quad E_2(z+\tau) = E_2(z)\,, \tag{1.11}
$$

- *Heat equation:*

$$
2\pi\imath\partial_\tau\phi(z,u) = \partial_z\partial_u\phi(z,u)\,, \tag{1.12}
$$

- *Derivatives:*

$$
\partial_u\phi(z,u) = \phi(z,u)(E_1(z+u) - E_1(u))\,, \tag{1.13}
$$

$$
\partial_z\phi(z,u) = \phi(z,u)(E_1(z+u) - E_1(z))\,, \tag{1.14}
$$

- *Fay (trisecant) identity [6]:*

$$
\phi(x,u)\phi(y,w) = \phi(x-y,u)\phi(y,u+w) + \phi(y-x,w)\phi(x,u+w)\,, \tag{1.15}
$$

- *Degenerated Fay identities:*

$$\phi(x,z)\phi(x,w) = \phi(x,z+w)(E_1(x)+E_1(z)+E_1(w)-E_1(x+z+w))\,, \tag{1.16}$$

or

$$\phi(x,z)\phi(y,z) = \phi(x+y,z)(E_1(x)+E_1(y)+E_1(z)-E_1(x+y+z))\,, \tag{1.17}$$

$$\phi(x,z)\phi(x,-z) = E_2(x)-E_2(z) = \wp(x)-\wp(z)\,. \tag{1.18}$$

- *Geometric interpretation:* The Kronecker function $\phi(z,u)$ is a section of the Poincaré bundle $\mathcal{P}$. It is a line bundle over $\Sigma_\tau \times \Sigma_\tau$, defined by the conditions (1.6), (1.7), (1.10), (1.11).

- *Green function:* The Kronecker function is the Green function for the operator $\bar\partial$ in the space of one forms $\mathcal{A}^{(1,0)}(\Sigma_\tau)$ with the boundary conditions (1.10) and (1.11):

$$\bar\partial\phi(z,u) = \delta^2(z,\bar z)\,. \tag{1.19}$$

Quantum R-matrices. Consider $\mathbb{Z}_N\times\mathbb{Z}_N$ (Baxter-Belavin's) elliptic R-matrix [2, 4] in the fundamental representation (see also [14]). It is defined via the finite-dimensional representation of the Heisenberg group:

$$Q,\Lambda\in \mathrm{Mat}(N,\mathbb{C}):\quad Q_{kl}=\delta_{kl}\exp(\frac{2\pi\imath}{N}k)\,,\quad \Lambda_{kl}=\delta_{k-l+1=0\,\mathrm{mod}N}\,,\quad k,l=1,...,N\,, \tag{1.20}$$

$$\exp(2\pi\imath\frac{\gamma_1\gamma_2}{N})Q^{\gamma_1}\Lambda^{\gamma_2}=\Lambda^{\gamma_2}Q^{\gamma_1}\,,\quad \gamma_1\,,\gamma_2\in\mathbb{Z}\,. \tag{1.21}$$

Introduce the sin-algebra basis in $\mathrm{Mat}(N,\mathbb{C})$:

$$T_\gamma := T_{\gamma_1\gamma_2} = \exp(\pi\imath\frac{\gamma_1\gamma_2}{N})Q^{\gamma_1}\Lambda^{\gamma_2}\,,\quad \gamma_1\,,\gamma_2=0,...,N-1\,. \tag{1.22}$$

The same definition is used for any $\gamma\in\mathbb{Z}^{\times 2}$. Then

$$T_\alpha T_\beta = \kappa_{\alpha,\beta}T_{\alpha+\beta}\,,\quad \kappa_{a,b}=\exp\left(\frac{\pi\imath}{N}(\beta_1\alpha_2-\beta_2\alpha_1)\right)\,, \tag{1.23}$$

where $\alpha+\beta=(\alpha_1+\beta_1,\alpha_2+\beta_2)$. The R-matrix is defined as

$$R^\hbar_{12}(u) = \sum_{\alpha\in\mathbb{Z}_N\times\mathbb{Z}_N}\varphi_\alpha(u,\omega_\alpha+\hbar)\,T_\alpha\otimes T_{-\alpha}\in \mathrm{Mat}(N,\mathbb{C})^{\otimes 2}\,, \tag{1.24}$$

where[1]

$$\varphi_\alpha(u,\omega_\alpha+\hbar)=\exp(2\pi\imath u\partial_\tau\omega_\alpha)\phi(u,\omega_\alpha+\hbar)\,,\quad \omega_\alpha=\frac{\alpha_1+\alpha_2\tau}{N}\,. \tag{1.25}$$

The $\mathbb{Z}_N\times\mathbb{Z}_N$ symmetry means that for $g=Q,\Lambda$

$$(g\otimes g)R^\hbar_{12}(u)(g^{-1}\otimes g^{-1}) = R^\hbar_{12}(u)\,. \tag{1.26}$$

[1]Here $\partial_\tau\omega_\alpha=\alpha_2/N$.

For $N=1$ the R-matrix (1.24) is the scalar Kronecker function $\phi(\hbar,u)$ (1.1). Notice that (1.24) is normalized in such a way that the unitarity condition acquires the form:

$$R^{\hbar}_{12}(u)R^{\hbar}_{21}(-u)=N^2\phi(N\hbar,u)\phi(N\hbar,-u)1\otimes 1=N^2(\wp(N\hbar)-\wp(u))1\otimes 1\,. \tag{1.27}$$

The latter can be considered as analogue of (1.18). Here $R_{21}(z)=P_{12}R_{12}(z)P_{12}$, where

$$P_{12}=\frac{1}{N}\sum_{\alpha}T_\alpha\otimes T_{-\alpha}=\sum_{i,j=1}^{N}E_{ij}\otimes E_{ji}\,,\quad (E_{ij})_{kl}=\delta_{ik}\delta_{jl} \tag{1.28}$$

is the permutation operator. We also use notation $R^{\hbar}_{ab}(z)$ which differs from (1.27) by $T^a_\alpha\otimes T^b_{-\alpha}=1\otimes ...1\otimes T_\alpha\otimes 1...1\otimes T_{-\alpha}\otimes 1...\otimes 1$ instead of $T_\alpha\otimes T_{-\alpha}$ (i.e. T_α and $T_{-\alpha}$ are in the a-th and b-th components). The number of components in the tensor product is an integer $\tilde N$. It means that $R^{\hbar}_{ab}$ is considered as an element of $\mathrm{Mat}(N,\mathbb{C})^{\otimes\tilde N}$, i.e. $N^{\tilde N}\times N^{\tilde N}$ matrix.

The properties and identities (1.8)-(1.17) have the following analogues for R-matrices:

- *Arguments symmetry:*

$$R^{\hbar}_{12}(z)=R^{\frac{z}{N}}_{12}(N\hbar)P_{12}\,, \tag{1.29}$$

- *Local expansion* in $\hbar$ is the classical limit:

$$R^{\hbar}_{12}(z)=\hbar^{-1}\,1\otimes 1+r_{12}(z)+\hbar\, m_{12}(z)+O(\hbar^2)\,, \tag{1.30}$$

where $r_{12}(z)$ is the classical (Belavin-Drinfeld [4]) r-matrix:

$$r_{12}(z)=E_1(z)\,1\otimes 1+\sum_{\alpha\neq 0}\varphi_\alpha(z)\,T_\alpha\otimes T_{-\alpha} \tag{1.31}$$

and

$$m_{12}(z)=\frac{E_1^2(z)-\wp(z)}{2}\,1\otimes 1+\sum_{\alpha\neq 0}\exp(2\pi\imath z\partial_\tau\omega_\alpha)\partial_u\phi(z,u)\mid_{u=\omega_\alpha}\,T_\alpha\otimes T_{-\alpha}\,. \tag{1.32}$$

Similarly to (1.7) we have:

$$r^2_{12}(z)-2m_{12}(z)=1\otimes 1\,N^2\wp(z)\,, \tag{1.33}$$

i.e. the quantum R-matrix is a matrix analogue of the Kronecker function (1.1) while the classical one is the analogue of the first Eisenstein function (1.2).

Expansion with respect to z (near $z=0$) is as follows:

$$R^{\hbar}_{12}(z)=\frac{NP_{12}}{z}+R^{\hbar,(0)}_{12}+O(z)\,, \tag{1.34}$$

where[2]

$$R^{\hbar,(0)}_{12}=\sum_{\alpha}T_\alpha\otimes T_{-\alpha}\,(E_1(\hbar+\omega_\alpha)+2\pi\imath\partial_\tau\omega_\alpha)\,. \tag{1.35}$$

[2] $R^{\hbar,(0)}_{12}$ appears as a part of the inverse inertia tensor for relativistic tops [9].

- *Residues*
$$\operatorname*{Res}_{\hbar=0} R^{\hbar}_{12}(z) = 1\otimes 1\,, \quad \operatorname*{Res}_{z=0} R^{\hbar}_{12}(z) = \operatorname*{Res}_{z=0} r_{12}(z) = N P_{12}\,, \tag{1.36}$$

- *Parity:*
$$R^{\hbar}_{12}(z) = -R^{-\hbar}_{21}(-z)\,, \quad r_{12}(z) = -r_{21}(-z)\,, \quad m_{12}(z) = m_{21}(-z)\,. \tag{1.37}$$
The R-matrix analogue of $E_2(u) = E_2(-u)$ (1.2) appears as $F^0_{12}(u) = -\partial_u r_{12}(u)$ (It is natural because $r_{12}(u)$ is the analogue of $E_1(u)$). The classical r-matrix is odd. Hence $F^0_{12}(u)$ is even matrix function. The same answer follows from the local expansions (1.7), (1.30): $E_2(u) = -\partial_u \phi(z,u)\mid_{z=0}$, then $-\partial_u R^z_{12}(u)\mid_{z=0} = -\partial_u r_{12}(u)$.

- *(Quasi)periodicity properties:*

$$R^{\hbar}_{12}(z+N\omega_\gamma) = \exp(-2\pi\imath N\hbar\,\partial_\tau\omega_\gamma)\,(T_\gamma^{-1}\otimes 1)R^{\hbar}_{12}(z)(T_\gamma\otimes 1)\,, \tag{1.38}$$
$$R^{\hbar+\omega_\gamma}_{12}(z) = \exp(-2\pi\imath z\partial_\tau\omega_\gamma)\,(T_\gamma^{-1}\otimes 1)R^{\hbar}_{12}(z)(1\otimes T_\gamma)\,. \tag{1.39}$$
In particular,
$$\begin{gathered} R^{\hbar}_{12}(z+1) = (Q^{-1}\otimes 1)R^{\hbar}_{12}(z)(Q\otimes 1)\,, \\ R^{\hbar}_{12}(z+\tau) = \exp(-2\pi\imath\hbar)\,(\Lambda^{-1}\otimes 1)R^{\hbar}_{12}(z)(\Lambda\otimes 1)\,, \end{gathered} \tag{1.40}$$
$$R^{\hbar+1}_{12}(z) = R^{\hbar}_{12}(z)\,, \quad R^{\hbar+\tau}_{12}(z) = \exp(-2\pi\imath z)\,R^{\hbar}_{12}(z)\,, \tag{1.41}$$

$$\begin{gathered} r_{12}(z+1) = (Q^{-1}\otimes 1)r_{12}(z)(Q\otimes 1)\,, \\ r_{12}(z+\tau) = (\Lambda^{-1}\otimes 1)r_{12}(z)(\Lambda\otimes 1) - 2\pi\imath\,1\otimes 1\,. \end{gathered} \tag{1.42}$$
Let us also rewrite (1.39) as follows:
$$R^{\hbar+1/N}_{ab}(z_a-z_b) = Q_a^{-1}R^{\hbar}_{ab}(z_a-z_b)\,Q_b\,, \tag{1.43}$$
$$R^{\hbar+\tau/N}_{ab}(z_a-z_b) = \exp(-2\pi\imath\frac{z_a-z_b}{N})\,\Lambda_a^{-1}R^{\hbar}_{ab}(z_a-z_b)\,\Lambda_b\,. \tag{1.44}$$
Recall now the R-matrix valued Lax matrix for $\mathrm{g}_{\tilde N}$ Calogero-Moser model [8]:

$$\mathcal{L}(\hbar) = \sum_{a,b=1}^{\tilde N} \tilde{\mathrm{E}}_{ab}\otimes\mathcal{L}_{ab}(\hbar)\,, \quad \mathcal{L}_{ab}(\hbar) = \delta_{ab}p_a\,1_a\otimes 1_b + \nu(1-\delta_{ab})R^{\hbar}_{ab}(z_a-z_b)\,. \tag{1.45}$$

where $\tilde{\mathrm{E}}_{ab}$ is the standard basis of $\mathrm{gl}_{\tilde N}$: $(\tilde{\mathrm{E}}_{ab})_{cd} = \delta_{ac}\delta_{bd}$, $a,b,c,d = 1...\tilde N$. Then it follows from (1.43)-(1.44) that
$$\begin{gathered} \mathcal{L}(\hbar+1/N) = \mathbf{Q}^{-1}\mathcal{L}(\hbar)\,\mathbf{Q}\,, \\ \mathcal{L}(\hbar+\tau/N) = \exp(-\mathbf{Z}/N)\,\mathbf{\Lambda}^{-1}\mathcal{L}(\hbar)\,\mathbf{\Lambda}\,\exp(\mathbf{Z}/N)\,, \end{gathered} \tag{1.46}$$
where
$$\mathbf{Q} = \bigoplus_{a=1}^{\tilde N} Q_a\,, \quad \mathbf{\Lambda} = \bigoplus_{a=1}^{\tilde N} \Lambda_a\,, \quad \mathbf{Z} = \bigoplus_{a=1}^{\tilde N} z_a 1_a \tag{1.47}$$
are block diagonal matrices. The number of blocks is $\tilde N\times\tilde N$, the size of a block is $N^{\tilde N}\times N^{\tilde N}$.

- *Heat equation:*

$$2\pi\imath\partial_\tau R^\hbar_{12}(z) = \partial_z\partial_\hbar R^\hbar_{12}(z)\,. \tag{1.48}$$

- *Derivatives*[3]*:*

$$\begin{array}{c}\displaystyle \partial_\hbar R^\hbar_{12}(z) = \frac{1}{2}\Big(\, r_{12}(z+N\hbar)\, R^\hbar_{12}(z) + R^\hbar_{12}(z)\, r_{12}(z-N\hbar)\,\Big) \\[2ex] \displaystyle +\frac{N}{2}\Big(E_1(z+N\hbar) - E_1(z-N\hbar) - 2E_1(N\hbar)\Big)R^\hbar_{12}(z)\,,\end{array} \tag{1.49}$$

$$\begin{array}{c}\displaystyle \partial_z R^\hbar_{12}(z) = \frac{1}{2N}\Big(\, r_{12}(z+N\hbar)\, R^\hbar_{12}(z) - R^\hbar_{12}(z)\, r_{12}(z-N\hbar)\,\Big) \\[2ex] \displaystyle +\frac{1}{2}\Big(E_1(z+N\hbar) + E_1(z-N\hbar) - 2E_1(z)\Big)R^\hbar_{12}(z)\,.\end{array} \tag{1.50}$$

- *The Fay identity in Mat*$(N,\mathbb{C})^{\otimes 3}$ *[1, 13, 8]:*

$$R^\hbar_{ab}R^{\hbar'}_{bc} = R^{\hbar'}_{ac}R^{\hbar-\hbar'}_{ab} + R^{\hbar'-\hbar}_{bc}R^\hbar_{ac}\,,\quad R^\hbar_{ab} = R^\hbar_{ab}(z_a - z_b)\,. \tag{1.51}$$

  Both parts of the identity are elements of $\mathrm{Mat}(N,\mathbb{C})^{\otimes 3}$. It was used in [8] for constructing higher-dimensional Lax pairs for Calogero-Moser models. Here we will prove another analogue of (1.15) – in $\mathrm{Mat}(N,\mathbb{C})^{\otimes 2}$.

- *The Fay identity in Mat*$(N,\mathbb{C})^{\otimes 2}$*:*

$$R^\hbar_{12}(z)R^{\hbar'}_{21}(-w) = \tag{1.52}$$

$$N\phi(N\hbar',\frac{z-w}{N}+\hbar'-\hbar)\, R^{\hbar-\hbar'}_{12}(z+N\hbar') - N\phi(N\hbar,\frac{z-w}{N}+\hbar'-\hbar)\, R^{\hbar-\hbar'}_{12}(w+N\hbar)$$

$$+N\phi(-w,\frac{z-w}{N}+\hbar'-\hbar)\, R^{\frac{z-w}{N}}_{12}(w+N\hbar) - N\phi(-z,\frac{z-w}{N}+\hbar'-\hbar)\, R^{\frac{z-w}{N}}_{12}(z+N\hbar')\,.$$

  The scalar analogue of this identity is obtained as follows: apply (1.15) (with $x=\hbar$, $y=\hbar'$) to $\phi(\hbar,z)\phi(\hbar',-w)$, and then apply (1.15) once again to the obtained r.h.s.. Then we get the scalar analogue of r.h.s. of (1.52).

- *Degenerated Fay identities in Mat*$(N,\mathbb{C})^{\otimes 3}$ *(1.51):*

$$R^\hbar_{ab}R^\hbar_{bc} = R^\hbar_{ac}r_{ab} + r_{bc}R^\hbar_{ac} - \partial_\hbar R^\hbar_{ac}\,, \tag{1.53}$$

$$R^\hbar_{ab}(z)R^{\hbar'}_{bc}(-z) = R^{\hbar',(0)}_{ac}R^{\hbar-\hbar'}_{ab}(z) + R^{\hbar'-\hbar}_{bc}(-z)R^{\hbar,(0)}_{ac} + NF^{\hbar'-\hbar}_{bc}(-z)P_{ac}\,, \tag{1.54}$$

  where $F^\hbar_{ab}(u) = \partial_u R^\hbar_{ab}(u)$ and $R^{\hbar,(0)}_{ab}$ is from (1.34)-(1.35).

[3] The identities for derivatives of R-matrix with respect to the Planck constant and spectral parameter were found in [3] and [15] respectively. Authors of [3, 15] used different normalization of the R-matrix.

- *Degenerated Fay identities in Mat$(N,\mathbb{C})^{\otimes 2}$ (1.52):*

$$R^{\hbar}_{12}(z)R^{\hbar}_{21}(-w)=N\phi(\frac{z-w}{N},N\hbar)\,(\,r_{12}(z+N\hbar)-r_{12}(w+N\hbar)\,)$$

$$+N\phi(\frac{w-z}{N},z)\,R_{12}^{\frac{z-w}{N}}(z+N\hbar)-N\phi(\frac{w-z}{N},w)\,R_{12}^{\frac{z-w}{N}}(w+N\hbar) \tag{1.55}$$

$$+N^2 1\otimes 1\,\phi(\frac{z-w}{N},N\hbar)\,(E_1(N\hbar)-E_1(N\hbar+\frac{z-w}{N}))\,,$$

and

$$R^{\hbar}_{12}(z)R^{\hbar'}_{21}(-z)=N\phi(\hbar'-\hbar,-z)\,(\,r_{12}(z+N\hbar)-r_{12}(z+N\hbar')\,)$$

$$-N\phi(\hbar'-\hbar,N\hbar)\,R_{12}^{\hbar-\hbar'}(z+N\hbar)+N\phi(\hbar'-\hbar,N\hbar')\,R_{12}^{\hbar-\hbar'}(z+N\hbar') \tag{1.56}$$

$$+N^2 1\otimes 1\,\phi(\hbar'-\hbar,-z)\,(E_1(z)-E_1(z+\hbar-\hbar'))\,.$$

- *Geometric interpretation.* Due to the quasi-periodicities (1.38)-(1.41) the R-matrix have the following geometrical interpretation. Let V_1 (V_2) be a rank N and degree one vector bundle over elliptic curve $\Sigma^{(1)}_\tau$ with coordinate z_1 ($\Sigma^{(2)}_\tau$ with coordinate z_2). Consider the bundle $V_1\boxtimes V_2$ over $\Sigma^{(1)}_\tau\times\Sigma^{(2)}_\tau$. Let $Aut_{\mathrm{PGL}(N)}(V_1\boxtimes V_2)$ be the automorphism group of the bundle (the gauge group). The sections $\Gamma(Aut_{\mathrm{PGL}(N)}(V_1\boxtimes V_2))$ depends only on the anti-diagonal $\tilde\Sigma_\tau$ of $\Sigma^{(1)}_\tau\times\Sigma^{(2)}_\tau$ with the coordinate $z=z_1-z_2$. Let $\tilde\Sigma'_\tau$ be the dual curve, $\hbar$ is the coordinate on $\tilde\Sigma'_\tau$ and $\mathcal{P}$ is the Poincaré bundle $\mathcal{P}$ over $\tilde\Sigma_\tau\times\tilde\Sigma'_\tau$ (1.5). Then the R-matrix (1.24) is a section

$$R^{\hbar}_{12}(z)\in\Gamma\left((Aut_{\mathrm{PGL}(N)}(V_1\boxtimes V_2))\otimes\mathcal{P}\right)\,.$$

- *Green function.* Similarly to (1.19) the R-matrix can be considered as the Green function of $\bar\partial$-operator:

$$\bar\partial R^{\hbar}_{12}(z)=NP_{12}\delta^2(z,\bar z)\,. \tag{1.57}$$

Properties (1.30)-(1.48) simply follows from their scalar counterparts except (1.33) which follows from the unitarity condition (1.27) in the classical limit (1.30). Identities for derivatives (1.49), (1.50) were obtained in [3, 15]. Degenerated Fay identities (1.53), (1.54) in Mat$(N,\mathbb{C})^{\otimes 3}$ follows from the nondegenerated one (1.51) and local expansions (1.30), (1.34).

Our main interest (in this paper) is the Fay identity in Mat$(N,\mathbb{C})^{\otimes 2}$ (1.52) and its degenerations (1.55), (1.56). We prove them below. The computational trick is based on the "arguments symmetry" property (1.29) and the scalar Fay identities (1.15)-(1.17).

Painlevé VI. As an application of the obtained formulae we construct higher-dimensional Lax pairs for the Painlevé VI equation. Denote the half-periods of the elliptic curve Σ_τ as

$$\{\Omega_a\,,\ a=0,1,2,3\}=\{0\,,\ \frac{1}{2}\,,\ \frac{1+\tau}{2}\,,\ \frac{\tau}{2}\}\,. \tag{1.58}$$

The Painlevé VI equation in the elliptic form [12] is

$$\frac{d^2u}{d\tau^2}=-\sum_{a=0}^{3}\nu_a^2\wp'(u+\Omega_a)\,. \tag{1.59}$$

Let N be an odd (positive) integer. Consider the following pair of block-matrices[4]:

$$L(\hbar)=\frac{1}{2}\frac{du}{d\tau}\begin{pmatrix}1\otimes 1 & 0\\ 0 & -1\otimes 1\end{pmatrix}+\sum_{a=0}^{3}\frac{\nu_a}{N\sqrt{-2}}\begin{pmatrix}0 & \mathcal{R}^{\hbar,a}_{12}(u)\\ \mathcal{R}^{\hbar,a}_{21}(-u) & 0\end{pmatrix} \tag{1.60}$$

$$M(\hbar)=\sum_{a=0}^{3}\frac{\nu_a}{N\sqrt{-2}}\begin{pmatrix}0 & \mathcal{F}^{\hbar,a}_{12}(u)\\ \mathcal{F}^{\hbar,a}_{21}(-u) & 0\end{pmatrix} \tag{1.61}$$

where

$$\begin{aligned}\mathcal{R}^{\hbar,a}_{12}(u)&=\exp(2\pi\imath N\hbar\,\partial_\tau\Omega_a)R^{\hbar}_{12}(u+N\Omega_a)\,,\\ \mathcal{R}^{\hbar,a}_{21}(-u)&=\exp(-2\pi\imath N\hbar\,\partial_\tau\Omega_a)R^{\hbar}_{21}(-u-N\Omega_a)\,,\end{aligned} \tag{1.62}$$

and

$$\begin{aligned}\mathcal{F}^{\hbar,a}_{12}(u)&=\exp(2\pi\imath N\hbar\,\partial_\tau\Omega_a)F^{\hbar}_{12}(u+N\Omega_a)\,,\\ \mathcal{F}^{\hbar,a}_{21}(-u)&=\exp(-2\pi\imath N\hbar\,\partial_\tau\Omega_a)F^{\hbar}_{21}(-u-N\Omega_a)\end{aligned} \tag{1.63}$$

with

$$F^{\hbar}_{ab}(u)=\partial_u R^{\hbar}_{ab}(u)\,. \tag{1.64}$$

The matrices $L(\hbar)\,,M(\hbar)\in\mathrm{Mat}(2,\mathbb{C})\otimes\mathrm{Mat}(N,\mathbb{C})^{\otimes 2}$. Their size equals $2N^2\times 2N^2$. The Painlevé VI equation (1.59) is equivalent to the monodromy preserving equation

$$\frac{d}{d\tau}L(\hbar)-\left(\frac{1}{2\pi\imath}\right)\frac{d}{d\hbar}M(\hbar)=[L(\hbar),M(\hbar)]\,, \tag{1.65}$$

where the Planck constant $\hbar$ plays the role of the spectral parameter (see [8]).

For $N=1$ the answer (1.60), (1.61) reproduces the elliptic 2×2 Lax pair proposed in [17].

The Lax pair (1.60), (1.61) works for even N's as well. But the Painlevé equation in this case has only one free constant:

$$\frac{d^2u}{d\tau^2}=-\nu^2\wp'(u)\,,\quad \nu^2=\sum_{a=0}^{3}\nu_a^2\,. \tag{1.66}$$

2 Kronecker double series and Baxter-Belavin R-matrix

Following idea suggested in [13] we derive here the Baxter-Belavin R-matrix as generalization of the Kronecker series.

R-matrix in Jacobi variables. Represent the elliptic curve Σ_τ (1.4) in the Jacobi form

$$C_q=\mathbb{C}/q^{\mathbb{Z}}\,,\quad q=\mathbf{e}(\tau)=\exp 2\pi\imath\tau\,.$$

Consider the product $C_q\times C_q$ with the coordinates $s=\mathbf{e}(u)$, $t=\mathbf{e}(z)$. Instead of the Kronecker function $\phi(z,u)$ we consider the distribution $g(s,t)$ on the space of the Laurent polynomials $\mathbb{C}[[s^{-1},t^{-1},s,t]]$. For $|q|<|t|<1$ it can be represented as the series

$$g(s,t|\,q)=\sum_{n\in\mathbb{Z}}\frac{t^n}{q^ns-1}\,. \tag{2.1}$$

[4]The coefficient $1/\sqrt{-2}$ gives the normalization of the constants as in (1.59).

If simultaneously $|q|<|s|<1$ then

$$g(s,t|\,q)=-g^{+}(s,t|\,q)+g^{-}(s,t|\,q)\,,\quad g^{+}(s,t|\,q)=\sum_{i,n\geq 0}s^{i}q^{in}t^{n}\,,\quad g^{-}(s,t|\,q)=\sum_{i,n<0}s^{i}q^{in}t^{n} \tag{2.2}$$

or

$$g(s,t|\,q)=1-\frac{1}{1-t}-\frac{1}{1-s}+g^{-}(s,t)-\sum_{i,n>0}s^{i}q^{in}t^{n}\,. \tag{2.3}$$

In the domain $|q|<|t|<1$ and $|q|<|s|<1$ we have

$$g(s,t|\,q)|_{s=\frac{1}{2\pi\imath}\ln u,\,t=\frac{1}{2\pi\imath}\ln z}=\phi(z,u)\,. \tag{2.4}$$

The distribution $g(s,t|\,q)$ has the properties analogous to (1.6)-(1.9). In particular,

$$g(s,t|\,q)=g(t,s|\,q)\,. \tag{2.5}$$

It follows from (2.2) that

$$g(s^{-1},t^{-1}|\,q)=-g(s,t|\,q)+\delta(t)+\delta(s)-2\,, \tag{2.6}$$

where $\delta(s)$ is the distribution on the space of the Laurent polynomials $\mathbb{C}[t,t^{-1}]=\{\psi(t)=\sum_l c_l t^l\}$, defined by the functional $\langle\delta,\psi\rangle=Res|_{t=0}\psi(t)$ and represented by the formal series

$$\delta(t)=\sum_{n\in\mathbb{Z}}t^{n}\,. \tag{2.7}$$

The analog of the quasiperiodic property (1.11) is the following. The distribution $g(s,t)$ is a solution of the difference equation on t (the Green function) variable

$$sg(s,tq|\,q)-g(s,t|\,q)=\delta(t)-1\,. \tag{2.8}$$

It defines the continuation of $g(s,t|\,q)$ from the annulus $|q|<|t|<1$ to $\mathbb{C}^*$. Due to (2.5) the similar equation can be written with respect to the s variable.

Let $\eta=\mathbf{e}(\hbar)$. The R-matrix (1.24) takes the following form in variables (s,t,η):

$$R^{\hbar}_{12}(s)=\sum_{\alpha\in\,\mathbb{Z}_N\times\mathbb{Z}_N}s^{\alpha_2/N}g(s,\omega_\alpha+\hbar)\,T_\alpha\otimes T_{-\alpha}= \tag{2.9}$$

$$\sum_{\alpha\in\,\mathbb{Z}_N\times\mathbb{Z}_N}s^{\alpha_2/N}\left(\sum_{m,n}\mathbf{e}(n\alpha_1/N)q^{n(m+\alpha_2/N)}\eta^{n}s^{m}\right)\,T_\alpha\otimes T_{-\alpha}\,.$$

It plays the role of the Green function for the difference operator

$$\eta\,(\Lambda\otimes 1)\,R^{\hbar}_{12}(sq)\,(\Lambda^{-1}\otimes 1)-R^{\hbar}_{12}(s)=(\delta(s)-1)P_{12}\,. \tag{2.10}$$

Kronecker double series [16]

The distribution $g(s,t|\,q)$ (and $\phi(z,u)$) can be represented as a Kronecker double series. Consider the lattice in $\mathbb{C}$

$$W=\{\gamma=m+n\tau\,,\ m,n\in\mathbb{Z}\}\,.$$

Represent the argument u of $\phi(z,u)$ as $u = u_1 + u_2\tau$ (u_1, u_2 are real), and let

$$\chi_u(\gamma) = \mathbf{e}(-mu_2 + nu_1)$$

be a character of the lattice W ($\chi_u(\gamma) : W \to S^1$), parameterized by $u \in \Sigma_\tau$. The Kronecker double series is defined as:

$$S(z,u|\,\tau) = \sum_{\gamma\in W} \frac{\chi_u(\gamma)}{z+\gamma}\,. \tag{2.11}$$

From the definition we find that

$$\begin{gathered} S(z+1,u|\,\tau) = \mathbf{e}(u_2)S(z,u|\,\tau)\,, \\ S(z+\tau,u|\,\tau) = \mathbf{e}(-u_1)S(z,u|\,\tau)\,. \end{gathered} \tag{2.12}$$

It was proved in [16] that $S(z,u|\,\tau)$ is related to the Kronecker function as

$$S(z,u|\,\tau) = \mathbf{e}(u_2 z)\phi(z,u)\,, \tag{2.13}$$

or in the Jacobi coordinates

$$S(t,s|\,q) = t^{u_2} g(s,t|\,q)\,. \tag{2.14}$$

Let us now pass to the R-matrix and describe it in terms of the Kronecker double series $S(z,u|\,\tau)$ (2.11).

Define the lattice W by the two generators $(\alpha_1/N + \hbar_1, (\alpha_2/N + \hbar_2)\tau)$, where $\hbar = \hbar_1 + \hbar_2\tau$, $\hbar_{1,2} \in \mathbb{R}$. The corresponding character of W is

$$\chi_{(m,n)}(\alpha,\hbar) = \mathbf{e}\left(-m(\alpha_2/N + \hbar_2) + n(\alpha_1/N + \hbar_1)\right)\,. \tag{2.15}$$

Then the R-matrix (1.24) is defined in terms of the Kronecker double series (2.11) as

$$R^\hbar_{12}(z) = \mathbf{e}(-\hbar_2 z) \sum_{(m,n)\in\mathbb{Z}\oplus\mathbb{Z}} \frac{\sum\limits_{\alpha\in\mathbb{Z}_N\times\mathbb{Z}_N} \chi_{(m,n)}(\alpha,\hbar)\, T_\alpha \otimes T_{-\alpha}}{z+m+n\tau}\,. \tag{2.16}$$

The quasi-periodicities (1.40), (1.41) now become evident. It follows from (2.13) that the singular behavior $z,\hbar \to 0$ of this representation is in agreement with (1.36).

We pass from $R^\hbar_{12}(z)$ to the modified matrix

$$\tilde{R}^\hbar_{12}(z) = \mathbf{e}(\hbar_2 z) R^\hbar_{12}(z)\,.$$

It satisfies the Yang-Baxter equation and has the quasi-periodicities

$$\tilde{R}^\hbar_{12}(z+1) = \mathbf{e}(\hbar_2)(Q^{-1}\otimes 1)\tilde{R}^\hbar_{12}(z)(Q\otimes 1)\,,$$

$$\tilde{R}^\hbar_{12}(z+\tau) = \mathbf{e}(\hbar_1)\,(\Lambda^{-1}\otimes 1)\tilde{R}^\hbar_{12}(z)(\Lambda\otimes 1)\,,$$

(compare with (1.40)). In contrast with (1.41) $\tilde{R}$ is not holomorphic in $\hbar$ and is double-periodic.

Remark 1 *The representation (2.16) means that the elliptic $\tilde{R}$-matrix is represented as the averaging of the Yang matrix $z^{-1}P_{12}$ along the lattice W twisted by the character (2.15).*

From (1.30) we also find the representation for the classical r-matrix:

$$r_{12}(z) = E_1(z)\, 1\otimes 1 + \sum_{m,n\in(\mathbb{Z}\oplus\mathbb{Z})\setminus(0,0)} \frac{\sum\limits_{\alpha\in\mathbb{Z}_N\times\mathbb{Z}_N} \chi_{(m,n)}(\alpha,0)\, T_\alpha\otimes T_{-\alpha}}{z+m+n\tau}$$

and

$$m_{12}(z) = \frac{E_1^2(z)-\wp(z)}{2}\, 1\otimes 1 + \sum_{m,n\in(\mathbb{Z}\oplus\mathbb{Z})\setminus(0,0)} \frac{\sum\limits_{\alpha\in\mathbb{Z}_N\times\mathbb{Z}_N} (z+m+n\bar{\tau})\chi_{(m,n)}(\alpha,0)\, T_\alpha\otimes T_{-\alpha}}{(z+m+n\tau)(\bar{\tau}-\tau)}\,.$$

3 Derivation of identities

Proposition 3.1 *The R-matrix (1.24) satisfies the arguments symmetry property (1.29).*

Proof: Using definitions (1.28) and (1.23) we have

$$\begin{aligned} R_{12}^{\frac{z}{N}}(N\hbar)P_{12} &= \frac{1}{N}\sum_{\alpha,\beta} T_\alpha T_\beta\otimes T_{-\alpha}T_{-\beta}\,\varphi_\alpha(N\hbar,\omega_\alpha+\frac{z}{N}) \\ &= \frac{1}{N}\sum_{\alpha,\beta}\kappa^2_{\alpha,\beta}\, T_{\alpha+\beta}\otimes T_{-\alpha-\beta}\,\varphi_\alpha(N\hbar,\omega_\alpha+\frac{z}{N})\,. \end{aligned} \tag{3.1}$$

Since $\kappa_{\alpha,\beta}=\kappa_{\alpha,\alpha+\beta}$, the property (1.29) is equivalent to the following set of N^2 identities:

$$\frac{1}{N}\sum_\alpha \kappa^2_{\alpha,\gamma}\,\varphi_\alpha(N\hbar,\omega_\alpha+\frac{z}{N}) = \varphi_\gamma(z,\omega_\gamma+\hbar)\,,\quad \forall\,\gamma\in\mathbb{Z}^{\times 2} \tag{3.2}$$

or

$$\frac{1}{N}\sum_\alpha \kappa^2_{\alpha,\gamma}\,\varphi_\alpha(z,\omega_\alpha+\hbar) = \varphi_\gamma(N\hbar,\omega_\gamma+\frac{z}{N})\,,\quad \forall\,\gamma\in\mathbb{Z}^{\times 2}\,. \tag{3.3}$$

The latter is verified by comparing residues. To do it we also need the relation for the sums of N-th roots of 1 (it also follows from $P_{12}^2=1$):

$$\sum_\alpha \kappa^2_{\alpha,\gamma} = N^2\delta_{\gamma,0}\,. \tag{3.4}$$

Let us calculate the residue of both parts of (3.2) at $\hbar=-\omega_\mu$. The answer for the r.h.s. is obviously $\delta_{\mu,\gamma}\exp(2\pi\imath\partial_\tau\omega_\gamma z)$ due to (1.8). For the l.h.s. we have:

$$\begin{aligned} &\operatorname*{Res}_{\hbar=-\omega_\mu}\frac{1}{N}\sum_\alpha\kappa^2_{\alpha,\gamma}\,\varphi_\alpha(N\hbar,\omega_\alpha+\frac{z}{N}) = \operatorname*{Res}_{\hbar=0}\frac{1}{N}\sum_\alpha\kappa^2_{\alpha,\gamma}\,\varphi_\alpha(N\hbar-N\omega_\mu,\omega_\alpha+\frac{z}{N}) \\ &\overset{(1.10),(1.11)}{=} \operatorname*{Res}_{\hbar=0}\frac{1}{N}\sum_\alpha\kappa^2_{\alpha,\gamma}\,\kappa^2_{\alpha,-\mu}\exp(2\pi\imath\partial_\tau\omega_\mu z)\,\varphi_\alpha(N\hbar,\omega_\alpha+\frac{z}{N}) \\ &\overset{(1.8)}{=} \frac{1}{N}\sum_\alpha\kappa^2_{\alpha,\gamma-\mu}\,\exp(2\pi\imath\partial_\tau\omega_\mu z)\frac{1}{N} \overset{(3.4)}{=} \delta_{\mu,\gamma}\exp(2\pi\imath\partial_\tau\omega_\mu z)\,.\ \blacksquare \end{aligned} \tag{3.5}$$

Proposition 3.2 *The R-matrix (1.24) satisfies the Fay identity (1.52) in Mat$(N,\mathbb{C})^{\otimes 2}$.*

Proof: Consider

$$R^{\hbar}_{12}(z)R^{\hbar'}_{21}(-w)=-\sum_{\alpha,\beta}\kappa^2_{\alpha,\beta}T_{\alpha+\beta}\otimes T_{-\alpha-\beta}\,\varphi_\alpha(z,\omega_\alpha+\hbar)\,\varphi_\beta(w,\omega_\beta-\hbar')= \tag{3.6}$$

Here we already used $R^{\hbar'}_{21}(-w)=-R^{-\hbar'}_{12}(w)$. Apply the Fay identity (1.15), then (3.3), and then (1.15) again:

$$\begin{aligned}&=-\sum_{\alpha,\beta}\kappa^2_{\alpha,\beta}T_{\alpha+\beta}\otimes T_{-\alpha-\beta}\,\varphi_\alpha(z-w,\omega_\alpha+\hbar)\,\varphi_{\alpha+\beta}(w,\omega_{\alpha+\beta}+\hbar-\hbar')\\&-\sum_{\alpha,\beta}\kappa^2_{\alpha,\beta}T_{\alpha+\beta}\otimes T_{-\alpha-\beta}\,\varphi_\beta(w-z,\omega_\beta-\hbar')\,\varphi_{\alpha+\beta}(z,\omega_{\alpha+\beta}+\hbar-\hbar')\end{aligned} \tag{3.7}$$

$$\begin{aligned}&=-N\sum_{\gamma}T_\gamma\otimes T_{-\gamma}\,\varphi_\gamma(N\hbar,\omega_\gamma+\frac{z-w}{N})\,\varphi_\gamma(w,\omega_\gamma+\hbar-\hbar')\\&+N\sum_{\gamma}T_\gamma\otimes T_{-\gamma}\,\varphi_\gamma(N\hbar',\omega_\gamma+\frac{z-w}{N})\,\varphi_\gamma(z,\omega_\gamma+\hbar-\hbar')\end{aligned} \tag{3.8}$$

$$\begin{aligned}=N\sum_{\gamma}T_\gamma\otimes T_{-\gamma}\Big(&-\phi(N\hbar,\frac{z-w}{N}+\hbar'-\hbar)\,\varphi_\gamma(w+N\hbar,\omega_\gamma+\hbar-\hbar')\\&-\phi(w,\hbar-\hbar'-\frac{z-w}{N})\,\varphi_\gamma(w+N\hbar,\omega_\gamma+\frac{z-w}{N})\\&+\phi(N\hbar',\frac{z-w}{N}+\hbar'-\hbar)\,\varphi_\gamma(z+N\hbar',\omega_\gamma+\hbar-\hbar')\\&+\phi(z,\hbar-\hbar'-\frac{z-w}{N})\,\varphi_\gamma(z+N\hbar',\omega_\gamma+\frac{z-w}{N})\Big)\,.\ \blacksquare\end{aligned} \tag{3.9}$$

Proposition 3.3 *The R-matrices (1.24) and (1.31) satisfies the degenerated Fay identities (1.55), (1.56) in Mat$(N,\mathbb{C})^{\otimes 2}$.*

Proof: We begin with (1.55). Consider

$$R^{\hbar}_{12}(z)R^{\hbar}_{21}(-w)=-\sum_{\alpha,\beta}\kappa^2_{\alpha,\beta}T_{\alpha+\beta}\otimes T_{-\alpha-\beta}\,\varphi_\alpha(z,\omega_\alpha+\hbar)\,\varphi_\beta(w,\omega_\beta-\hbar)\,. \tag{3.10}$$

Subdivide it into two parts: $\sum_{\alpha,\beta}=\sum_{\alpha\neq-\beta}+\sum_{\alpha=-\beta}$. The first part is transformed as in the previous Proposition (via (1.15), then (3.3), and then (1.15) again)

$$\begin{aligned}\sum_{\alpha\neq-\beta}=&-\sum_{\alpha\neq-\beta}\kappa^2_{\alpha,\beta}T_{\alpha+\beta}\otimes T_{-\alpha-\beta}\,\varphi_\alpha(z-w,\omega_\alpha+\hbar)\,\varphi_{\alpha+\beta}(w,\omega_{\alpha+\beta})\\&-\sum_{\alpha\neq-\beta}\kappa^2_{\alpha,\beta}T_{\alpha+\beta}\otimes T_{-\alpha-\beta}\,\varphi_\beta(w-z,\omega_\beta-\hbar)\,\varphi_{\alpha+\beta}(z,\omega_{\alpha+\beta})\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&= \ldots = -N\phi(\frac{z-w}{N}, N\hbar) \sum_{\gamma\neq 0} T_\gamma \otimes T_{-\gamma}\, \varphi_\gamma(w+N\hbar, \omega_\gamma) \\
&-N\phi(\frac{w-z}{N}, w) \sum_{\gamma\neq 0} T_\gamma \otimes T_{-\gamma}\, \varphi_\gamma(w+N\hbar, \omega_\gamma + \frac{z-w}{N}) \\
&+N\phi(\frac{z-w}{N}, N\hbar) \sum_{\gamma\neq 0} T_\gamma \otimes T_{-\gamma}\, \varphi_\gamma(z+N\hbar, \omega_\gamma) \\
&+N\phi(\frac{w-z}{N}, z) \sum_{\gamma\neq 0} T_\gamma \otimes T_{-\gamma}\, \varphi_\gamma(z+N\hbar, \omega_\gamma + \frac{z-w}{N})
\end{aligned}
\tag{3.12}
$$

By adding (and subtracting) scalar terms ($1\otimes 1$) to each line one obtains the first and the second lines of (1.55). The input to the scalar part should be summed up together with

$$
\begin{aligned}
\sum_{\alpha=-\beta} &= 1\otimes 1 \sum_\alpha \varphi_\alpha(z, \omega_\alpha + \hbar)\, \varphi_\alpha(-w, \omega_\alpha + \hbar) \\
&\overset{(1.17)}{=} 1\otimes 1 \sum_\alpha \varphi_\alpha(z-w, \omega_\alpha + \hbar)\, (E_1(z) - E_1(w) + E_1(\hbar+\omega_\alpha) - E_1(z-w+\hbar+\omega_\alpha))\,.
\end{aligned}
\tag{3.13}
$$

The latter expression is transformed via (3.3) for $\gamma = 0$

$$
\sum_\alpha \varphi_\alpha(z-w, \omega_\alpha + \hbar) = N\phi(N\hbar, \frac{z-w}{N})
$$

and its derivative (1.13), (1.14) with respect to $\hbar$:

$$
\begin{aligned}
&\sum_\alpha \varphi_\alpha(z-w, \omega_\alpha + \hbar)\, (E_1(z-w+\hbar+\omega_\alpha) - E_1(\hbar+\omega_\alpha)) \\
&\quad = N^2 \phi(N\hbar, \frac{z-w}{N}) \left(E_1(N\hbar + \frac{z-w}{N}) - E_1(N\hbar) \right)\,.
\end{aligned}
$$

This finishes the proof of (1.55). The identity (1.56) can be derived similarly. Equivalently, (1.56) follows from (1.55) by using the properties (1.29) and (1.37). ■

4 Higher-dimensional elliptic Lax pairs for Painlevé VI

Different types of matrix-valued Lax pairs for Painlevé equations are known (see e.g. [7, 5, 10]). In this section we construct R-matrix valued generalization of the elliptic 2×2 Lax pair suggested in [17].

Proposition 4.1 *The Painlevé VI equation in the elliptic form (1.59) is equivalent to the monodromy preserving equation (1.65) with the Lax pair (1.60)-(1.64) and the elliptic R-matrix (1.24) for odd N.*

Proof is similar to the one given in [17] for the scalar ($N = 1$) case. First, notice that $\frac{d}{d\tau} L(\hbar) = \frac{du}{d\tau}\partial_u L(\hbar) + \partial_\tau L(\hbar)$, where the last term is the derivative by explicit dependence on τ. It is canceled out by $\frac{1}{2\pi\imath}\frac{d}{d\hbar} M(\hbar)$ due to the heat equation (1.48) $2\pi\imath\, \partial_\tau \mathcal{R}^{\hbar,a}_{bc}(u) = \partial_\hbar \mathcal{F}^{\hbar,a}_{bc}(u)$.

Denote

$$L^a = \begin{pmatrix} 0 & \mathcal{R}^{\hbar,a}_{12}(u) \\ \mathcal{R}^{\hbar,a}_{21}(-u) & 0 \end{pmatrix}, \quad M^a = \begin{pmatrix} 0 & \mathcal{F}^{\hbar,a}_{12}(u) \\ \mathcal{F}^{\hbar,a}_{21}(-u) & 0 \end{pmatrix} \tag{4.1}$$

The main statement which we need to verify is that for $a \neq b$

$$[L^a, M^b] + [L^b, M^a] = 0\,, \tag{4.2}$$

i.e. the input to $[L(\hbar), M(\hbar)]$ comes only from $[L^a, M^a]$. Indeed, it follows from the unitarity condition (1.27) that

$$\mathcal{R}^{\hbar,a}_{12}(u)\mathcal{R}^{\hbar,a}_{21}(-u) = R^{\hbar}_{12}(u + N\Omega_a)R^{\hbar}_{21}(-u - N\Omega_a) = N^2(\wp(N\hbar) - \wp(u + N\Omega_a))\,. \tag{4.3}$$

Differentiating (4.3) with respect to u we get

$$\mathcal{F}^{\hbar,a}_{12}(u)\mathcal{R}^{\hbar,a}_{21}(-u) - \mathcal{R}^{\hbar,a}_{12}(u)\mathcal{F}^{\hbar,a}_{21}(-u) = -N^2\wp'(u + N\Omega_a)\,. \tag{4.4}$$

This identity provides the equation of motion. Notice that in order to have all four constants N should be odd since $\wp'(u+N\Omega_a) = \wp'(u+\Omega_a)$ in this case. If N is even then $\wp'(u+N\Omega_a) = \wp'(u)$, and we have only one constant as in (1.66).

To prove (4.2) let us recall that in the scalar case this followed from

$$\begin{gathered} \varphi_a(\hbar, u + \Omega_a)f_b(\hbar, -u - \Omega_b) - f_b(\hbar, u + \Omega_b)\varphi_a(\hbar, -u - \Omega_a) \\ \varphi_b(\hbar, u + \Omega_b)f_a(\hbar, -u - \Omega_a) - f_a(\hbar, u + \Omega_a)\varphi_b(\hbar, -u - \Omega_b) = 0\,, \end{gathered} \tag{4.5}$$

where

$$f_a(z, u + \Omega_a) = \exp(2\pi\imath\partial_\tau\Omega_a\hbar)\partial_w\phi(\hbar, w)\mid_{w=u+\Omega_a}$$

is the scalar analogue of $\mathcal{F}^{\hbar,a}_{12}(u)$. The identity (4.5) appears from (1.16) and (1.10)-(1.11) as follows:

$$\begin{gathered} \varphi_a(\hbar, u + \Omega_a)\varphi_b(\hbar, -u - \Omega_b) + \varphi_b(\hbar, u + \Omega_b)\varphi_a(\hbar, -u - \Omega_a) = \\ \varphi_{a+b}(\hbar, \Omega_a + \Omega_b)\,(2E_1(\hbar) - E_1(\hbar + \Omega_a - \Omega_b) - E_1(\hbar + \Omega_b - \Omega_a))\,. \end{gathered} \tag{4.6}$$

The r.h.s. of (4.6) is independent of u. The derivative of (4.6) with respect to u gives (4.5).

Similarly to (4.6) it follows from the degenerated Fay identity (1.55) that

$$\begin{gathered} \mathcal{R}^{\hbar,a}_{12}(u)\mathcal{R}^{\hbar,b}_{21}(-u) + \mathcal{R}^{\hbar,b}_{12}(u)\mathcal{R}^{\hbar,a}_{21}(-u) \\ = N^2\, 1 \otimes 1\, \varphi_{a+b}(N\hbar, \Omega_a + \Omega_b)\,(2E_1(N\hbar) - E_1(N\hbar + \Omega_a - \Omega_b) - E_1(N\hbar + \Omega_b - \Omega_a))\,. \end{gathered} \tag{4.7}$$

It can be verified directly using (1.10)-(1.11) which can be re-written as

$$\phi(z, w + \Omega_a) = \exp(-2\pi\imath z\partial_\tau\Omega_a)\phi(z, w - \Omega_a)\,.$$

The r.h.s. of (4.7) is scalar and independent of u. The derivative of (4.7) with respect to u gives

$$\mathcal{F}^{\hbar,a}_{12}(u)\mathcal{R}^{\hbar,b}_{21}(-u) - \mathcal{R}^{\hbar,a}_{12}(u)\mathcal{F}^{\hbar,b}_{21}(-u) + \mathcal{F}^{\hbar,b}_{12}(u)\mathcal{R}^{\hbar,a}_{21}(-u) - \mathcal{R}^{\hbar,b}_{12}(u)\mathcal{F}^{\hbar,a}_{21}(-u) = 0\,. \tag{4.8}$$

This identity underlies (4.2). ■

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